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The integrated inventory model with the transportation cost and two-level trade credit in supply chain management☆

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ABSTRACT

This paper takes the transportation cost into account to develop the new supplier–retailer inventory model under the condition that both supplier and retailer have adopted the two-level trade credit policy. Moreover, this paper presents the integrated total profit per unit time $\Pi(n, T)$ of two decision variables n (the number of shipments from supplier to retailer per production run, a positive integer) and T (retailer's replenishment cycle length). The main purpose of this paper not only derives the closed-form formulations for the optimal solution (n^*, T^*) of $\Pi(n, T)$ but also simplifies the algorithm to determine the optimal solution described by Su et al. (2007) [36]. Finally, numerical examples are used to compare with those by Su et al. (2007) [36].

1. Introduction

The integrated inventory models usually have the advantage of reducing total cost. In the modern global competitive market, the supplier and retailer should be treated as strategic partners in the supply chain with a long-term cooperative relationship. Goyal [1] was probably the first researcher to develop the seller–customer inventory model. Recently, numerous researchers developed integrated inventory models under various assumptions. For example, Rau and Ouyang [2] presented an integrated production–inventory policy under a finite planning horizon and a linear trend in demand. Ho et al. [3] developed an integrated supplier–buyer inventory model with the assumption that the market demand is sensitive to the retail price and the supplier adopts a trade credit policy. Ouyang et al. [4] explored an optimization approach for joint pricing and ordering problem in an integrated inventory system. Huang et al. [5,6] considered integrated vendor–buyer inventory models with order-processing cost reduction and permissible delay in payments. Subsequently, Chung and Liao [7] revealed the simplified solution algorithm for an integrated supplier–buyer inventory model with two-part trade credit in a supply chain system. Many related articles can be found in the comprehensive review paper of the joint economic lot-sizing problem discussed in [8].

Trade credit represents one of the most flexible sources of short-term financing available to firms principally because it arises spontaneously with the firm's purchases. Goyal [9] developed the economic order quantity model under conditions of permissible delay in payments. He assumed that the supplier would offer the retailer a delay period but the retailer would not offer the trade period to customers. That is one level of trade credit. Recently, Huang [10] and Teng and Goyal [11] extended Goyal [9] to provide a fixed trade credit period M between the supplier and the retailer and a trade credit period N between the retailer and the customer. That is two-level trade credit. The key points of differences between Huang [10] and Teng and Goyal [11] can be explained as follows:

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(A) From the viewpoint of Huang [10]: in [10], if a customer buys one item from the retailer at time t belonging to $[0, N]$, then the customer will have a trade credit period $N - t$ and make the payment at time N. So, a retailer allows a maximal trade credit period N for customers to settle the account. In fact, the trade credit periods offered by a retailer to customers are different. The customer's trade credit period N offered by a retailer in [10] should mean the due day which customers make their payments to the retailer. Basically, the viewpoint of Huang[10] can be used in the market of credit cards. The inventory models under two levels of trade credit policy from the viewpoint of Huang [10] have been found in many articles, such as [12–24]

(B) From the viewpoint of Teng and Goyal [11]: in [11], if a customer buys one item from the retailer at time t belonging to $[0, T]$, then the customer will have a trade credit period N and make the payment at time $N + t$. So, a retailer allows each customer with the same trade period N. The viewpoint of Teng and Goyal [11] can be used in the general business transactions. The inventory models under two levels of trade credit policy from the viewpoint of Teng and Goyal [11] have been found in many articles, such as [25–32].

Su et al. presented a stylized model to determine the optimal strategy for the integrated supplier–retailer inventory model under the condition that both the supplier and retailer adopted two-level trade credit policy. The transportation cost was not considered in their model. However, in a pioneering effort, Baumol and Vinod [33] suggested the inclusion of transportation costs in a comprehensive inventory-theoretic model. Recently, Ouyang et al. [34], Huang et al. [6] and Teng et al. [35] discussed integrated supplier–retailer inventory models incorporating the trade credit and transportation cost. From the viewpoint of practice, there exists the motivation to generalize [36] to consider transportation cost in the new integrated model for establishing the integrated total profit per unit time $\prod(n, T)$ of two decision variables n (the number of shipment per production run from the supplier to the retailer) and T (the replenishment cycle length). This paper uses the calculus approach not only to derive closed-form formulations for the optimal solution (n^*, T^*) but also to simplify the algorithm to locate the optimal solution (n^*, T^*) described in [36]. Finally, the numerical examples are used to compare with those in [36].

2. Notations and assumptions

The following notations and assumptions are adopted throughout the paper:

Notations:

R: supplier's production rate.
S_1: supplier's setup cost per setup.
S_2: retailer's ordering cost per order.
F: the transportation cost per delivery.
r_1: supplier's holding cost rate excluding interest charges.
r_2: retailer's holding cost rate excluding interest charges.
c: supplier's production cost per unit.
v: the unit price charged by the supplier to the retailer.
p: the unit retailer price charged by the retailer to customers, where $p > v > c$.
t_1: retailer's credit period offered by the supplier per order.
t_2: customer's credit period offered by the retailer, where $t_2 \leq t_1$.
I_{1p}: supplier's capital opportunity cost per dollar per unit time.
I_{2p}: retailer's capital opportunity cost per dollar per unit time.
I_{2e}: retailer's interest earned per dollar per unit time.
Q: retailer's order quantity per order (decision variable).
T: retailer's replenishment cycle length (decision variable).
n: number of shipments from supplier to retailer per production run, a positive integer (decision variable).
$TVP(n, T)$: the supplier's expected total profit per unit time.
$TBP(T)$: the retailer's expected total profit per unit time.
$\prod(n, T)$: the channel's expected total profit per unit time.

Assumptions:

(1) The inventory system consists of single-supplier, single-retailer and multiple customers.
(2) Only one type of item is considered.
(3) Shortages are not allowed.
(4) The supplier offers a credit period t_1 to the retailer and the retailer offers a credit period t_2 to each customer, where $t_2 < t_1$.
(5) The market demand rate $D = D(t_2) = D_0 e^{\delta t_2}$, where D_0 is an average market demand rate per unit time when the retailer does not offer a credit period to customers, and $\delta \geq 0$ is a constant. For notational simplicity, D and $D(t_2)$ will be used interchangeably in this paper.
(6) $R > D$.

(7) The production and shipping policies are described as follows: the retailer orders quantity $Q(=DT)$ per order and the supplier manufactures in batches of size nQ. During the production period, once the first Q units are produced, the supplier delivers them to the retailer and then continuously makes a delivery every Q/D unit of time until the supplier's inventory level falls to zero.

3. Model formulation

3.1. Supplier's expected total profit per unit time

Throughout each production run, the supplier manufactures in batches of size nQ. As the first Q units have been produced, the supplier distributes them to the retailer directly, after which time the supplier will make the delivery on average Q/D units of time. Adopting the same techniques of arguments as those in [36], we can demonstrate that the supplier's expected total profit per unit time is expressed as

$$\begin{aligned} TVP(n,T) &= \text{sales revenue} - \text{production cost} - \text{setup cost} - \text{holding cost} - \text{opportunity cost} \\ &= Dv - Dc - \frac{S_1}{nT} - \frac{cDT(r_1+I_{1p})[(n-1)(R-D)+D]}{2R} - vI_{1p}Dt_1. \end{aligned} \tag{1}$$

3.2. Retailer's expected total profit per unit time

Throughout each production run, the supplier manufactures in batches of size nQ. Hence,

$$\text{the expected number of production runs per unit time } = \frac{D}{nQ}. \tag{2}$$

Recall that n denotes the number of shipments from supplier to retailer per production run. Therefore,

$$\text{the number of shipments from supplier to retailer per unit time} = (n)\left(\frac{D}{nQ}\right) = \frac{D}{Q} = \frac{1}{T}, \tag{3}$$

and

$$\text{the transportation cost per unit time} = \frac{F}{T}. \tag{4}$$

According to Eq. (4), adopting the same techniques of arguments as those in [36], we can demonstrate that the expected total profit per unit time for the retailer is

$$TBP(T) = \begin{cases} TBP_1(T) & \text{if } 0 < T \le t_1 - t_2, \quad \text{(a)} \\ TBP_2(T) & \text{if } t_1 - t_2 \le T \le t_1, \quad \text{(b)} \\ TBP_3(T) & \text{if } t_1 \le T, \quad \text{(c)} \end{cases} \tag{5}$$

where

$$\begin{aligned} TBP_1(T) &= \text{sales revenue} - \text{purchasing cost} - \text{ordering cost} - \text{transportation cost} \\ &\quad - \text{holding cost} + \text{interest earned} \\ &= Dp - Dv - \frac{S_2+F}{T} - \frac{vr_2DT}{2} + pI_{2e}D\left(t_1 - t_2 - \frac{T}{2}\right), \end{aligned} \tag{6}$$

$$\begin{aligned} TBP_2(T) &= \text{sales revenue} - \text{purchasing cost} - \text{ordering cost} - \text{transportation cost} \\ &\quad - \text{holding cost} + \text{interest earned} - \text{capital opportunity cost} \\ &= Dp - Dv - \frac{S_2+F}{T} - \frac{vr_2DT}{2} + \frac{pI_{2e}D(t_1-t_2)^2}{2T} - \frac{pI_{2p}D(T+t_2-t_1)^2}{2T}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} TBP_3(T) &= \text{sales revenue} - \text{purchasing cost} - \text{ordering cost} - \text{transportation cost} \\ &\quad - \text{holding cost} + \text{interest earned} - \text{capital opportunity cost} \\ &= Dp - Dv - \frac{S_2+F}{T} - \frac{vr_2DT}{2} + \frac{pI_{2e}D(t_1-t_2)^2}{2T} - \frac{I_{2p}D[pt_2^2 + 2pt_2(T-t_1) + v(T-t_1)^2]}{2T}. \end{aligned} \tag{8}$$

3.3. The integrated total profit per unit time

If the supplier and retailer want to establish a long-term strategic partnership and contract to commit the relationship, then they will jointly determine the best policy for the whole supply chain system. Based on the above statement, the integrated total profit per unit time $\prod(n, T)$ can be expressed as follows:

$$\prod(n,T) = \begin{cases} \prod_1(n,T) & \text{if } 0 < T \le t_1 - t_2, \quad \text{(a)} \\ \prod_2(n,T) & \text{if } t_1 - t_2 \le T \le t_1, \quad \text{(b)} \\ \prod_3(n,T) & \text{if } T \ge t_1, \quad \text{(c)} \end{cases} \tag{9}$$

where

$$\begin{aligned} \prod_1(n,T) &= TVP(n,T) + TBP_1(T) \\ &= Dp - Dc - \frac{vr_2DT}{2} + pI_{2e}D\left(t_1 - t_2 - \frac{T}{2}\right) \\ &\quad - \frac{cDT(r_1 + I_{1p})}{2R}[(n-1)(R-D)+D] - I_{1p}vDt_1 - \frac{1}{T}\left(\frac{S_1}{n} + S_2 + F\right), \end{aligned} \tag{10}$$

$$\begin{aligned} \prod_2(n,T) &= TVP(n,T) + TBP_2(T) \\ &= Dp - Dc - \frac{vr_2DT}{2} + \frac{pI_{2e}D(t_1-t_2)^2}{2T} - \frac{pI_{2p}D(T+t_2-t_1)^2}{2T} \\ &\quad - \frac{cDT(r_1 + I_{1p})}{2R}[(n-1)(R-D)+D] - I_{1p}vDt_1 - \frac{1}{T}\left(\frac{S_1}{n} + S_2 + F\right), \end{aligned} \tag{11}$$

and

$$\begin{aligned} \prod_3(n,T) &= TVP(n,T) + TBP_3(T) \\ &= Dp - Dc - \frac{vr_2DT}{2} + \frac{pI_{2e}D(t_1-t_2)^2}{2T} - \frac{I_{2p}D}{2T}[pt_2^2 + 2pt_2(T-t_1) + v(T-t_1)^2] \\ &\quad - \frac{cDT(r_1 + I_{1p})}{2R}[(n-1)(R-D)+D] - I_{1p}vDt_1 - \frac{1}{T}\left(\frac{S_1}{n} + S_2 + F\right) \end{aligned} \tag{12}$$

It is obvious that $\prod_1(n, t_1 - t_2) = \prod_2(n, t_1 - t_2)$ and $\prod_2(n, t_1) = \prod_3(n, t_1)$. Hence, for any given n, $\prod(n, T)$ is a continuous function on $T > 0$. The problem now is to determine the optimal replenishment cycle length of the retailer, T^*, and the optimal shipment number, n^*, from the supplier to the retailer per production run, such that $\prod(n^*, T^*)$ is the maximum value.

Remark 1. If $F = 0$, then [36] is a special case of this paper.

4. The concavity of $\prod_i(n, T)(i = 1, 2, 3)$

Eqs. (10)–(12) yield that

$$\frac{\partial \prod_1(n,T)}{\partial T} = \frac{-D}{2R}\{R(vr_2 + pI_{2e}) + A[(n-1)(R-D)+D]\} + \frac{1}{T^2}\left(\frac{S_1}{n} + S_2 + F\right), \tag{13}$$

$$\begin{aligned} \frac{\partial \prod_2(n,T)}{\partial T} &= \frac{-D}{2R}\left\{R(vr_2 + pI_{2p}) + A[(n-1)(R-D)+D]\right\} \\ &\quad + \frac{Dp(I_{2p} - I_{2e})(t_1-t_2)^2}{2T^2} + \frac{1}{T^2}\left(\frac{S_1}{n} + S_2 + F\right) \end{aligned} \tag{14}$$

$$\begin{aligned} \frac{\partial \prod_3(n,T)}{\partial T} &= \frac{-D}{2R}\left\{Rv(r_2 + I_{2p}) + A[(n-1)(R-D)+D]\right\} \\ &\quad + \frac{Dp(I_{2p} - I_{2e})(t_1-t_2)^2}{2T^2} + \frac{D(v-p)I_{2p}t_1^2}{2T^2} + \frac{1}{T^2}\left(\frac{S_1}{n} + S_2 + F\right), \end{aligned} \tag{15}$$

$$\frac{\partial^2 \prod_1(n,T)}{\partial T^2} = -\frac{2}{T^3}\left(\frac{S_1}{n} + S_2 + F\right) < 0, \tag{16}$$

$$\frac{\partial^2 \prod_2(n,T)}{\partial T^2} = -\frac{1}{T^3}\left\{Dp(I_{2p}-I_{2e})(t_1-t_2)^2 + 2\left(\frac{S_1}{n}+S_2+F\right)\right\}, \tag{17}$$

and

$$\frac{\partial^2 \prod_3(n,T)}{\partial T^2} = -\frac{1}{T^3}\left\{D[p(I_{2p}-I_{2e})(t_1-t_2)^2 + (v-p)I_{2p}t_1^2] + 2\left(\frac{S_1}{n}+S_2+F\right)\right\}. \tag{18}$$

Eqs. (13)–(15) show that

$$\begin{aligned}\frac{\partial \prod_1(n,t_1-t_2)}{\partial T} &= \frac{\partial \prod_2(n,t_1-t_2)}{\partial T}\\ &= \frac{\frac{-D(t_1-t_2)^2}{2R}\{R(vr_2+pI_{2e})+A[(n-1)(R-D)+D]\}+\left(\frac{S_1}{n}+S_2+F\right)}{(t_1-t_2)^2},\end{aligned} \tag{19}$$

and

$$\begin{aligned}\frac{\partial \prod_2(n,t_1)}{\partial T} &= \frac{\partial \prod_3(n,t_1)}{\partial T}\\ &= \frac{\frac{-D}{2R}\left\{t_1^2\left\{R(vr_2+pI_{2p})+A\left[(n-1)(R-D)+D\right]\right\}-Rp(I_{2p}-I_{2e})(t_1-t_2)^2\right\}+\left(\frac{S_1}{n}+S_2+F\right)}{t_1^2}.\end{aligned} \tag{20}$$

Let

$$G_1(n) = \frac{D(t_1-t_2)^2}{2R}\{R(vr_2+pI_{2e})+A[(n-1)(R-D)+D]\} - \left(\frac{S_1}{n}+S_2+F\right), \tag{21}$$

$$G_2(n) = \frac{D}{2R}\left\{t_1^2\left\{R(vr_2+pI_{2p})+A[(n-1)(R-D)+D]\right\} - Rp(I_{2p}-I_{2e})(t_1-t_2)^2\right\} - \left(\frac{S_1}{n}+S_2+F\right), \tag{22}$$

$$H_1(n) = Dp(I_{2p}-I_{2e})(t_1-t_2)^2 + 2\left(\frac{S_1}{n}+S_2+F\right), \tag{23}$$

and

$$H_2(n) = D[p(I_{2p}-I_{2e})(t_1-t_2)^2 + (v-p)I_{2p}t_1^2] + 2\left(\frac{S_1}{n}+S_2+F\right). \tag{24}$$

Then, $G_2(n) > G_1(n)$ if $n \geq 1$.

Solving

$$\frac{\partial \prod_i(n,T)}{\partial T} = 0 \quad (i=1,2,3), \tag{25}$$

yields

$$T_{1,n} = \sqrt{\frac{\frac{2R}{D}\left(\frac{S_1}{n}+S_2+F\right)}{R(vr_2+pI_{2e})+A[(n-1)(R-D)+D]}}, \tag{26}$$

$$T_{2,n} = \sqrt{\frac{\frac{2R}{D}\left(\frac{S_1}{n}+S_2+F\right)+Rp(I_{2p}-I_{2e})(t_1-t_2)^2}{R(vr_2+pI_{2p})+A[(n-1)(R-D)+D]}} \quad \text{if } H_1(n) > 0, \tag{27}$$

and

$$T_{3,n} = \sqrt{\frac{\frac{2R}{D}\left(\frac{S_1}{n}+S_2+F\right)+R[p(I_{2p}-I_{2e})(t_1-t_2)^2+(v-p)I_{2p}t_1^2]}{Rv(r_2+I_{2p})+A[(n-1)(R-D)+D]}} \quad \text{if } H_2(n) > 0, \tag{28}$$

respectively, as the solutions of Eq. (25) if $n \geq 1$. Furthermore, if $T_{i,n}$ exists,

$$\frac{\partial \prod_i(n,T)}{\partial T} \begin{cases} > 0 & \text{if } 0 < T < T_{i,n}, \quad \text{(a)} \\ = 0 & \text{if } T = T_{i,n}, \quad \text{(b)} \\ < 0 & \text{if } T > T_{i,n}. \quad \text{(c)} \end{cases} \tag{29}$$

Eq. (29)(a)–(c) imply that $\prod_i(n,T)$ is increasing on $(0, T_{i,n}]$ and decreasing on $[T_{i,n}, \infty)$ for $i = 1, 2, 3$, and any given $n \geq 1$. So, we have the following results.

Lemma 1. (A) *$\prod_1(n,T)$ is concave on $T > 0$ and $T_{1,n}$ exists for all $n \geq 1$.*

(B) *If $H_1(n) > 0$, then $\prod_2(n,T)$ is concave on $T > 0$ and $T_{2,n}$ exists; otherwise, if $H_1(n) \leq 0$, then $\prod_2(n,T)$ is decreasing on $T > 0$ and $T_{2,n}$ does not exist for all $n \geq 1$.*

(C) *If $H_2(n) > 0$, then $\prod_3(n,T)$ is concave on $T > 0$ and $T_{3,n}$ exists; otherwise, if $H_2(n) \leq 0$, then $\prod_3(n,T)$ is decreasing on $T > 0$ and $T_{3,n}$ does not exist for all $n \geq 1$.*

Proof. (A) Eqs. (16) and (26) imply Lemma 1(A) holds.

(B) If $H_1(n) > 0$, Eqs. (17) and (27) imply that $\prod_2(n,T)$ is concave on $T > 0$ and $T_{2,n}$ exists. Furthermore, if $H_1(n) \leq 0$, Eq. (14) yields

$$\frac{\partial \prod_2(n,T)}{\partial T} < 0 \quad \text{if } T > 0. \tag{30}$$

Eqs. (27) and (30) reveal that $\prod_2(n,T)$ is decreasing on $T > 0$ and $T_{2,n}$ does not exist for all $n \geq 1$.

(C) If $H_2(n) > 0$, Eqs. (18) and (28) imply that $\prod_3(n,T)$ is concave on $T > 0$ and $T_{3,n}$ exists. Furthermore, if $H_2(n) \leq 0$, Eq. (15) yields

$$\frac{\partial \prod_3(n,T)}{\partial T} < 0 \quad \text{if } T > 0. \tag{31}$$

Eqs. (28) and (31) reveal that $\prod_3(n,T)$ is decreasing on $T > 0$ and $T_{3,n}$ does not exist for all $n \geq 1$.

Incorporating the above arguments, we have completed the proof of Lemma 1. □

Lemma 2. (A) *For a fixed n, if $G_1(n) \leq 0$, then,*

(i) *$H_1(n) > 0$,*

(ii) *$\prod_2(n,T)$ is concave on $T > 0$,*

(iii) *$T_{2,n}$ exists.*

(B) *For a fixed n, if $G_2(n) \leq 0$, then,*

(i) *$H_2(n) > 0$,*

(ii) *$\prod_3(n,T)$ is concave on $T > 0$,*

(iii) *$T_{3,n}$ exists.*

Proof. (A): For a fixed n, if $G_1(n) \leq 0$, then

$$\frac{S_1}{n} + S_2 + F \geq \frac{D(t_1 - t_2)^2}{2} (vr_2 + pI_{2e}). \tag{32}$$

Eq. (32) implies

$$H_1(n) > D(t_1 - t_2)^2 \left(vr_2 + pI_{2p}\right) > 0. \tag{33}$$

Eqs. (33), (17) and (27) demonstrate that Lemma 2(A) holds.

(B) For a fixed n, if $G_2(n) \leq 0$, then

$$\frac{S_1}{n} + S_2 + F \geq \frac{D}{2} \left\{t_1^2(vr_2 + pI_{2p}) - p(I_{2p} - I_{2e})(t_1 - t_2)^2\right\}. \tag{34}$$

Eq. (34) implies

$$H_2(n) > Dt_1^2 v(I_{2p} + r_2). \tag{35}$$

Eqs. (35), (18) and (28) demonstrate that Lemma 2(B) holds.

Incorporating (A) and (B), we have completed the proof of Lemma 2. □

5. Theorem for the optimal replenishment cycle length $T^{(n)}$ of $\prod(n, T)$ when n is fixed

Let $T^{(n)}$ denote the optimal replenishment cycle length of $\prod(n, T)$for a fixed n. Thus, we have the following result.

Theorem 1. *For any given $n \geq 1$, we have*
(A) *if $G_1(n) \geq 0$, then $T^{(n)} = T_{1,n}$,*
(B) *if $G_1(n) < 0 \leq G_2(n)$, then $T^{(n)} = T_{2,n}$,*
(C) *if $G_2(n) < 0$, then $T^{(n)} = T_{3,n}$.*

Proof. (A) If $G_1(n) \geq 0$, then $G_2(n) \geq G_1(n) \geq 0$. With Lemma 1 and Eqs. (29)(a)–(c), we have

(i) $\prod_1(n, T)$ is increasing on $(0, T_{1,n}]$ and decreasing on $[T_{1,n}, t_1 - t_2]$.
(ii) $\prod_2(n, T)$ is decreasing on $[t_1 - t_2, t_1]$.
(iii) $\prod_3(n, T)$ is decreasing on $[t_1, \infty)$.

Combining Eq. (9)(a)–(c) and (i)–(iii), we conclude $T^{(n)} = T_{1,n}$.

(B) If $G_1(n) < 0 \leq G_2(n)$, with Lemmas 1, 2(A) and Eqs. (29)(a)–(c), we have

(iv) $\prod_1(n, T)$ is increasing on $(0, t_1 - t_2]$.
(v) $\prod_2(n, T)$ is increasing on $[t_1 - t_2, T_{2,n}]$ and decreasing on $[T_{2,n}, t_1]$.
(vi) $\prod_3(n, T)$ is decreasing on $[t_1, \infty)$.

Combining Eq. (9)(a)–(c) and (iv)–(vi), we conclude $T^{(n)} = T_{2,n}$.

(C) If $G_2(n) < 0$, then $G_1(n) < G_2(n) < 0$. With Lemmas 1, 2(A) and (B) and Eqs. (29)(a)–(c), we have

(vii) $\prod_1(n, T)$ is increasing on $(0, t_1 - t_2]$.
(viii) $\prod_2(n, T)$ is increasing on $[t_1 - t_2, t_1]$.
(ix) $\prod_3(n, T)$ is increasing on $[t_1, T_{3,n}]$ and decreasing on $[T_{3,n}, \infty)$.

Combining Eqs. (9)(a)–(c) and (vii)–(ix), we conclude $T^{(n)} = T_{3,n}$.

Incorporating the above arguments, we have completed the proof of Theorem 1. □

Remark 2. If $F = 0$,based on the above arguments, we have the following observations:

(P1) Eq. (4.11) in [36] is valid if and only if $H_1(n) > 0$. If $H_1(n) \leq 0$, then $T_{2,n}$ does not exist. Moreover, if $G_1(n) < 0 \leq G_2(n)$, the process of the proof of Theorem 4.1(b) in [36] does not explain why $T_{2,n}$ exists. Lemma 2(A) in this paper complements the shortcoming of Theorem 4.1(b) in [36].
(P2) Eq. (4.13) in [36] is valid if and only if $H_2(n) > 0$. If $H_2(n) \leq 0$, then $T_{3,n}$ does not exist. Moreover, if $G_2(n) < 0$, the process of the proof of Theorem 4.1(c) in [36] does not explain why $T_{3,n}$ exists. Lemma 2(B) in this paper complements the shortcoming of Theorem 4.1(c) in [36].

Combining (P1) and (P2), we conclude that the proofs of Theorem 4.1(b) and (c) in [36] are not complete from the viewpoint of logic. So, Theorem 1 in this paper gives the complete proofs for Theorem 4.1(b) and (c) in [36].

6. The closed-form formulations for the optimal solution (n^*, T^*) of $\prod(n, T)$

Let (n^*, T^*) denote the optimal solution of $\prod(n, T)$. Then

$$\prod(n^*, T^*) = \max_{n\geq 1} \left\{\prod(n, T^{(n)})\right\}. \tag{36}$$

Furthermore, Lemmas 1 and 2 reveal that if $T_{i,n}$ exists, then $\prod_i(n, T)$ is concave on $T > 0$ for any given $n \geq 1$ and $i = 1, 2, 3$. Let

$$TC_i(n) = \prod_i(n, T_{i,n}) \quad (i = 1, 2, 3). \tag{37}$$

If we treat n as a continuous variable, taking the derivative of Eq. (37) with respect to n yields that

$$\frac{dTC_1(n)}{dn} = K_1(n)\frac{dT_{1,n}}{\partial n} + \frac{A(R-D)(S_2+F)\left\{-n^2 + \frac{S_1\{R(vr_2+pI_{2e})+A(2D-R)\}}{A(R-D)(S_2+F)}\right\}}{n^2 T_{1,n}\{R(vr_2+pI_{2e}) + A[(n-1)(R-D)+D]\}}, \tag{38}$$

$$\begin{aligned}\frac{dTC_2(n)}{dn} &= K_2(n)\frac{dT_{2,n}}{\partial n} + \frac{DA(R-D)\left[\frac{2(S_2+F)}{D} + p(I_{2p}-I_{2e})(t_1-t_2)^2\right]}{2n^2 T_{2,n}\left\{R(vr_2+pI_{2p}) + A[(n-1)(R-D)+D]\right\}}\\ &\quad\times\left\{-n^2 + \frac{2S_1[R(vr_2+pI_{2p})+A(2D-R)]}{DA(R-D)\left[\frac{2(S_2+F)}{D} + p(I_{2p}-I_{2e})(t_1-t_2)^2\right]}\right\},\end{aligned} \tag{39}$$

and

$$\frac{dTC_3(n)}{dn} = K_3(n)\frac{dT_{3,n}}{\partial n} + \frac{DA(R-D)\left[\frac{2(S_2+F)}{D} + p(I_{2p}-I_{2e})(t_1-t_2)^2 + (v-p)I_{2p}t_1^2\right]}{2n^2T_{3,n}\left\{Rv(r_2+I_{2p}) + A[(n-1)(R-D)+D]\right\}} \times \left\{-n^2 + \frac{2S_1[Rv(r_2+I_{2p}) + A(2D-R)]}{DA(R-D)\left[\frac{2(S_2+F)}{D} + p(I_{2p}-I_{2e})(t_1-t_2)^2 + (v-p)I_{2p}t_1^2\right]}\right\}, \tag{40}$$

where

$$A = c(r_1 + I_{1p}), \tag{41}$$

$$K_1(n) = \frac{-D}{2R}\{R(vr_2 + pI_{2e}) + A[(n-1)(R-D)+D]\} + \frac{1}{(T_{1,n})^2}\left(\frac{S_1}{n} + S_2 + F\right), \tag{42}$$

$$K_2(n) = \frac{-D}{2R}\left\{R(vr_2 + pI_{2p}) + A[(n-1)(R-D)+D]\right\} + \frac{Dp(I_{2p}-I_{2e})(t_1-t_2)^2 + 2\left(\frac{S_1}{n} + S_2 + F\right)}{2(T_{2,n})^2}, \tag{43}$$

and

$$lK_3(n) = \frac{-D}{2R}\left\{Rv(r_2+I_{2p}) + A[(n-1)(R-D)+D]\right\} + \frac{Dp(I_{2p}-I_{2e})(t_1-t_2)^2 + D(v-p)I_{2p}t_1^2 + 2\left(\frac{S_1}{n} + S_2 + F\right)}{2(T_{3,n})^2}. \tag{44}$$

Eq. (25) implies $K_i(n) = 0$ if $T_{i,n}$ exist for $i = 1, 2, 3$ and any given $n \geq 1$.

Therefore, Eqs. (38)–(40) can be simplified as follows:

$$\frac{dTC_1(n)}{dn} = \frac{A(R-D)(S_2+F)\left\{-n^2 + \frac{S_1\{R(vr_2+pI_{2e})+A(2D-R)\}}{A(R-D)(S_2+F)}\right\}}{n^2T_{1,n}\{R(vr_2+pI_{2e}) + A[(n-1)(R-D)+D]\}}, \tag{45}$$

$$\frac{dTC_2(n)}{dn} = \frac{DA(R-D)\left[\frac{2(S_2+F)}{D} + p(I_{2p}-I_{2e})(t_1-t_2)^2\right]}{2n^2T_{2,n}\left\{R(vr_2+pI_{2p}) + A[(n-1)(R-D)+D]\right\}} \times \left\{-n^2 + \frac{2S_1[R(vr_2+pI_{2p}) + A(2D-R)]}{DA(R-D)\left[\frac{2(S_2+F)}{D} + p(I_{2p}-I_{2e})(t_1-t_2)^2\right]}\right\} \tag{46}$$

and

$$\frac{dTC_3(n)}{dn} = \frac{DA(R-D)\left[\frac{2(S_2+F)}{D} + p(I_{2p}-I_{2e})(t_1-t_2)^2 + (v-p)I_{2p}t_1^2\right]}{2n^2T_{3,n}\left\{Rv(r_2+I_{2p}) + A[(n-1)(R-D)+D]\right\}} \times \left\{-n^2 + \frac{2S_1[Rv(r_2+I_{2p}) + A(2D-R)]}{DA(R-D)\left[\frac{2(S_2+F)}{D} + p(I_{2p}-I_{2e})(t_1-t_2)^2 + (v-p)I_{2p}t_1^2\right]}\right\}. \tag{47}$$

Next, we let

$$E_1 = \sqrt{\frac{S_1[R(vr_2+pI_{2e}) + A(2D-R)]}{A(R-D)(S_2+F)}}, \tag{48}$$

$$E_2 = \sqrt{\frac{2S_1[R(vr_2+pI_{2p}) + A(2D-R)]}{DA(R-D)\left[\frac{2(S_2+F)}{D} + p(I_{2p}-I_{2e})(t_1-t_2)^2\right]}}. \tag{49}$$

$$E_3 = \sqrt{\frac{2S_1[Rv(r_2+I_{2p}) + A(2D-R)]}{DA(R-D)\left[\frac{2(S_2+F)}{D} + p(I_{2p}-I_{2e})(t_1-t_2)^2 + (v-p)I_{2p}t_1^2\right]}}. \tag{50}$$

$$P_1 = R(vr_2 + pI_{2e}) + A(2D-R), \tag{51}$$

$$P_2 = R(vr_2 + pI_{2p}) + A(2D-R), \tag{52}$$

and

$$P_3 = Rv(r_2 + I_{2p}) + A(2D - R). \tag{53}$$

Eqs. (45)–(53) imply that the following results hold.

Lemma 3. *Suppose that $T_{i,n}$ exists for $i = 1, 2, 3$. Hence,*

(IA) *if $P_1 \leq 0$, then $TC_1(n)$ is decreasing on $n \geq 1$.*
(IB) *if $P_1 > 0$, then $TC_1(n)$ is increasing on $(0, E_1]$ and decreasing on $[E_1, \infty)$.*
(IIA) *if $P_2 \leq 0$, then $TC_2(n)$ is decreasing on $n \geq 1$.*
(IIB) *if $P_2 > 0$, then $TC_2(n)$ is increasing on $(0, E_2]$ and decreasing on $[E_2, \infty)$.*
(IIIA) *if $P_3 \leq 0$, then $TC_3(n)$ is decreasing on $n \geq 1$.*
(IIIB) *if $P_3 > 0$, then $TC_3(n)$ is increasing on $(0, E_3]$ and decreasing on $[E_3, \infty)$.*

Proof. (IA): If $P_1 \leq 0$, then Eq. (45) implies

$$\frac{dTC_1(n)}{dn} < 0 \quad \text{if } n \geq 1. \tag{54}$$

Eq. (54) illustrates that $TC_1(n)$ is decreasing on $n \geq 1$.

(IB): If $P_1 > 0$, then Eq. (45) implies

$$\frac{dTC_1(n)}{dn} \begin{cases} > 0 & \text{if } 0 < n < E_1, & \text{(a)} \\ = 0 & \text{if } n = E_1, & \text{(b)} \\ < 0 & \text{if } n > E_1. & \text{(c)} \end{cases} \tag{55}$$

Eqs. (55)(a)–(c) illustrate that $TC_1(n)$ is increasing on $(0, E_1]$ and decreasing on $[E_1, \infty)$.

(IIA)–(IIB): The same techniques of arguments as those in [IA, IB] can be applied on Eq. (46) to demonstrate that both (IIA) and (IIB) hold.

(IIIA)–(IIIB): The same techniques of arguments as those in [IA, IB] can be applied on Eq. (47) to demonstrate that both (IIIA) and (IIIB) hold.

Incorporating (IA)–(IIIB), we have completed the proof of Lemma 3. □

Subsequently, Eqs. (21) and (22) imply

$$\frac{dG_1(n)}{dn} = \frac{D(t_1 - t_2)^2}{2R} A(R - D) + \frac{S_1}{n^2} > 0, \tag{56}$$

and

$$\frac{dG_2(n)}{dn} = \frac{D}{2R} t_1^2 A(R - D) + \frac{S_1}{n^2} > 0. \tag{57}$$

Eqs. (56) and (57) demonstrate that both $G_1(n)$ and $G_2(n)$ are increasing on $n \geq 1$. Furthermore, Eqs. (21) and (22) reveal that

$$G_1(1) = \frac{D(t_1 - t_2)^2}{2R} [R(vr_2 + pI_{2e}) + AD] - (S_1 + S_2 + F), \tag{58}$$

$$G_2(1) = \frac{D}{2R} \left\{ t_1^2 [R(vr_2 + pI_{2p}) + AD] - Rp(I_{2p} - I_{2e})(t_1 - t_2)^2 \right\} - (S_1 + S_2 + F). \tag{59}$$

Furthermore, we let $n_1^* = \lfloor E_1 \rfloor$, $n_2^* = \lfloor E_2 \rfloor$ and $n_3^* = \lfloor E_3 \rfloor$ where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Then, we have the following results.

Theorem 2. *Case* (I)*: suppose that $G_1(1) \geq 0$. Hence,*

(A) *if $P_1 \leq 0$, then*

$$\prod(n^*, T^*) = \prod(1, T_{1,1}) = \prod_1 (1, T_{1,1}). \tag{60}$$

(B) *if $P_1 > 0$, then*

$$\prod(n^*, T^*) = \left\{ \begin{matrix} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \end{matrix} \right\}. \tag{61}$$

Case (II)*: suppose that* $G_1(1) < 0$ *and* $G_2(1) \geq 0$*. Let* n_{01} *be the smallest positive integer such that* $G_1(n_{01}) \geq 0$ *and* $n_{01} > 1$*. So, there are four sub-cases to occur:*

(A) *If* $P_1 \leq 0$ *and* $P_2 \leq 0$*, then*

$$\prod(n^*, T^*) = \left\{ \begin{matrix} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (1, T_{2,1}) \end{matrix} \right\}. \tag{62}$$

(B) *Suppose that* $P_1 > 0$ *and* $P_2 > 0$*. Hence,*

(B1) *if* $n_1^* < n_{01}$ *and* $n_2^* < n_{01} - 1$*, then*

$$\prod(n^*, T^*) = \left\{ \begin{matrix} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_2^*, T_{2,n_2^*}) \\ \prod_2 (n_2^* + 1, T_{2,n_2^*+1}) \end{matrix} \right\}. \tag{63}$$

(B2) *if* $n_1^* \geq n_{01}$ *and* $n_2^* < n_{01} - 1$*, then*

$$\prod(n^*, T^*) = \left\{ \begin{matrix} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_2 (n_2^*, T_{2,n_2^*}) \\ \prod_2 (n_2^* + 1, T_{2,n_2^*+1}) \end{matrix} \right\}. \tag{64}$$

(B3) *if* $n_1^* < n_{01}$ *and* $n_2^* \geq n_{01} - 1$*, then*

$$\prod(n^*, T^*) = \max \left\{ \begin{matrix} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_{01} - 1, T_{2,n_{01}-1}) \end{matrix} \right\}. \tag{65}$$

(B4) *if* $n_1^* \geq n_{01}$ *and* $n_2^* \geq n_{01} - 1$*, then*

$$\prod(n^*, T^*) = \max \left\{ \begin{matrix} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_2 (n_{01} - 1, T_{2,n_{01}-1}) \end{matrix} \right\}. \tag{66}$$

(C) *Suppose that* $P_1 \leq 0$ *and* $P_2 > 0$*. Hence,*

(C1) *if* $n_2^* < n_{01} - 1$*, then*

$$\prod(n^*, T^*) = \max \left\{ \begin{matrix} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_2^*, T_{2,n_2^*}) \\ \prod_2 (n_2^* + 1, T_{2,n_2^*+1}) \end{matrix} \right\}. \tag{67}$$

(C2) *if* $n_2^* \geq n_{01} - 1$*, then*

$$\prod(n^*, T^*) = \max \left\{ \begin{matrix} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_{01} - 1, T_{2,n_{01}-1}) \end{matrix} \right\}. \tag{68}$$

(D) *Suppose that* $P_1 > 0$ *and* $P_2 \leq 0$*. Hence,*

(D1) *if* $n_1^* < n_{01}$*, then*

$$\prod(n^*, T^*) = \max \left\{ \begin{matrix} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (1, T_{2,1}) \end{matrix} \right\}. \tag{69}$$

(D2) *if* $n_1^* \geq n_{01}$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_2 (1, T_{2,1}) \end{array} \right\}. \tag{70}$$

Case (III)*: suppose that* $G_1(1) < 0$ *and* $G_2(1) < 0$*. Let* n_{02} *be the smallest positive integer such that* $G_2(n_{02}) \geq 0$ *and* $n_{01} \geq n_{02} > 1$*. So, there are two situations* $(\mathrm{S}-1)$*:* $n_{01} > n_{02}$ *and* $(\mathrm{S}-2)$*:* $n_{01} = n_{02}$ *to occur:*

(S-1): $n_{01} > n_{02}$.

(A) *If* $P_1 \leq 0$ *and* $P_3 < P_2 \leq 0$*, then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_{02}, T_{2,n_{02}}) \\ \prod_3 (1, T_{3,1}) \end{array} \right\}. \tag{71}$$

(B) *Suppose that* $P_1 > 0$ *and* $P_2 > P_3 > 0$*. Hence,*

(B1) *if* $n_1^* < n_{01}$, $n_2^* < n_{02}$ *and* $n_3^* < n_{02} - 1$*, then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_{02}, T_{2,n_{02}}) \\ \prod_3 (n_3^*, T_{3,n_3^*}) \\ \prod_3 (n_3^* + 1, T_{3,n_3^*+1}) \end{array} \right\}. \tag{72}$$

(B2) *if* $n_1^* \geq n_{01}$, $n_2^* < n_{02}$ *and* $n_3^* < n_{02} - 1$*, then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_2 (n_{02}, T_{2,n_{02}}) \\ \prod_3 (n_3^*, T_{3,n_3^*}) \\ \prod_3 (n_3^* + 1, T_{3,n_3^*+1}) \end{array} \right\}. \tag{73}$$

(B3) *if* $n_1^* < n_{01}$, $n_{02} \leq n_2^* < n_{01} - 1$ *and* $n_3^* < n_{02} - 1$*, then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_2^*, T_{2,n_2^*}) \\ \prod_2 (n_2^* + 1, T_{2,n_2^*+1}) \\ \prod_3 (n_3^*, T_{3,n_3^*}) \\ \prod_3 (n_3^* + 1, T_{3,n_3^*+1}) \end{array} \right\}. \tag{74}$$

(B4) *if* $n_1^* \geq n_{01}, n_{02} \leq n_2^* < n_{01} - 1$ *and* $n_3^* < n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_2 (n_2^*, T_{2,n_2^*}) \\ \prod_2 (n_2^* + 1, T_{2,n_2^*+1}) \\ \prod_3 (n_3^*, T_{3,n_3^*}) \\ \prod_3 (n_3^* + 1, T_{3,n_3^*+1}) \end{Bmatrix}. \tag{75}$$

(B5) *if* $n_1^* < n_{01}, n_2^* \geq n_{01} - 1$ *and* $n_3^* < n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_{01} - 1, T_{2,n_{01}-1}) \\ \prod_3 (n_3^*, T_{3,n_3^*}) \\ \prod_3 (n_3^* + 1, T_{3,n_3^*+1}) \end{Bmatrix}. \tag{76}$$

(B6) *if* $n_1^* \geq n_{01}, n_2^* \geq n_{01} - 1$ *and* $n_3^* < n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_2 (n_{01} - 1, T_{2,n_{01}-1}) \\ \prod_3 (n_3^*, T_{3,n_3^*}) \\ \prod_3 (n_3^* + 1, T_{3,n_3^*+1}) \end{Bmatrix}. \tag{77}$$

(B7) *if* $n_1^* < n_{01}, n_2^* < n_{02}$ *and* $n_3^* \geq n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_{02}, T_{2,n_{02}}) \\ \prod_3 (n_{02} - 1, T_{3,n_{02}-1}) \end{Bmatrix}. \tag{78}$$

(B8) *if* $n_1^* \geq n_{01}, n_2^* < n_{02}$ *and* $n_3^* \geq n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_2 (n_{02}, T_{2,n_{02}}) \\ \prod_3 (n_{02} - 1, T_{3,n_{02}-1}) \end{Bmatrix}. \tag{79}$$

(B9) *if* $n_1^* < n_{01}$, $n_{02} \le n_2^* < n_{01} - 1$ *and* $n_3^* \ge n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_2^*, T_{2,n_2^*}) \\ \prod_2 (n_2^* + 1, T_{2,n_2^*+1}) \\ \prod_3 (n_{02} - 1, T_{3,n_{02}-1}) \end{Bmatrix}. \tag{80}$$

(B10) *if* $n_1^* \ge n_{01}$, $n_{02} \le n_2^* < n_{01} - 1$ *and* $n_3^* \ge n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_2 (n_2^*, T_{2,n_2^*}) \\ \prod_2 (n_2^* + 1, T_{2,n_2^*+1}) \\ \prod_3 (n_{02} - 1, T_{3,n_{02}-1}) \end{Bmatrix}. \tag{81}$$

(B11) *if* $n_1^* < n_{01}$, $n_2^* \ge n_{01} - 1$ *and* $n_3^* \ge n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_{01} - 1, T_{2,n_{01}-1}) \\ \prod_3 (n_{02} - 1, T_{3,n_{02}-1}) \end{Bmatrix}. \tag{82}$$

(B12) *if* $n_1^* \ge n_{01}$, $n_2^* \ge n_{01} - 1$ *and* $n_3^* \ge n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_2 (n_{01} - 1, T_{2,n_{01}-1}) \\ \prod_3 (n_{02} - 1, T_{3,n_{02}-1}) \end{Bmatrix}. \tag{83}$$

(C) *Suppose that* $P_1 \le 0$, $P_2 > 0$ *and* $P_3 \le 0$. *Hence,*

(C1) *if* $n_2^* < n_{02}$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_{02}, T_{2,n_{02}}) \\ \prod_3 (1, T_{3,1}) \end{Bmatrix}. \tag{84}$$

(C2) *if* $n_{02} \le n_2^* < n_{01} - 1$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_2^*, T_{2,n_2^*}) \\ \prod_2 (n_2^* + 1, T_{2,n_2^*+1}) \\ \prod_3 (1, T_{3,1}) \end{Bmatrix}. \tag{85}$$

(C3) *if* $n_2^* \geq n_{01} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_{01} - 1, T_{2,n_{01}-1}) \\ \prod_3 (1, T_{3,1}) \end{array} \right\}. \tag{86}$$

(D) *Suppose that* $P_1 \leq 0$ *and* $P_2 > P_3 > 0$. *Hence,*

(D1) *if* $n_2^* < n_{02}$ *and* $n_3^* < n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_{02}, T_{2,n_{02}}) \\ \prod_3 (n_3^*, T_{3,n_3^*}) \\ \prod_3 (n_3^* + 1, T_{3,n_3^*+1}) \end{array} \right\}. \tag{87}$$

(D2) *if* $n_{02} \leq n_2^* < n_{01} - 1$ *and* $n_3^* < n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_2^*, T_{2,n_2^*}) \\ \prod_2 (n_2^* + 1, T_{2,n_2^*+1}) \\ \prod_3 (n_3^*, T_{3,n_3^*}) \\ \prod_3 (n_3^* + 1, T_{3,n_3^*+1}) \end{array} \right\}. \tag{88}$$

(D3) *if* $n_2^* \geq n_{01} - 1$ *and* $n_3^* < n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_{01} - 1, T_{2,n_{01}-1}) \\ \prod_3 (n_3^*, T_{3,n_3^*}) \\ \prod_3 (n_3^* + 1, T_{3,n_3^*+1}) \end{array} \right\}. \tag{89}$$

(D4) *if* $n_2^* < n_{02}$ *and* $n_3^* \geq n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_{02}, T_{2,n_{02}}) \\ \prod_3 (n_{02} - 1, T_{3,n_{02}-1}) \end{array} \right\}. \tag{90}$$

(D5) *if* $n_{02} \leq n_2^* < n_{01} - 1$ *and* $n_3^* \geq n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_2^*, T_{2,n_2^*}) \\ \prod_2 (n_2^* + 1, T_{2,n_2^*+1}) \\ \prod_3 (n_{02} - 1, T_{3,n_{02}-1}) \end{array} \right\}. \tag{91}$$

(D6) *if* $n_2^* \geq n_{01} - 1$ *and* $n_3^* \geq n_{02} - 1$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1(n_{01}, T_{1,n_{01}}) \\ \prod_2(n_{01} - 1, T_{2,n_{01}-1}) \\ \prod_3(n_{02} - 1, T_{3,n_{02}-1}) \end{Bmatrix}. \tag{92}$$

(E) *Suppose that* $P_1 > 0$ *and* $P_3 < P_2 \leq 0$. *Hence*

(E1) *if* $n_1^* < n_{01}$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1(n_{01}, T_{1,n_{01}}) \\ \prod_2(n_{02}, T_{2,n_{02}}) \\ \prod_3(1, T_{3,1}) \end{Bmatrix}. \tag{93}$$

(E2) *if* $n_1^* \geq n_{01}$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1(n_1^*, T_{1,n_1^*}) \\ \prod_1(n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_2(n_{02}, T_{2,n_{02}}) \\ \prod_3(1, T_{3,1}) \end{Bmatrix}. \tag{94}$$

(F) *Suppose that* $P_1 > 0$, $P_2 > 0$ *and* $P_3 \leq 0$. *Hence,*

(F1) *if* $n_1^* < n_{01}$ *and* $n_2^* < n_{02}$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1(n_{01}, T_{1,n_{01}}) \\ \prod_2(n_{02}, T_{2,n_{02}}) \\ \prod_3(1, T_{3,1}) \end{Bmatrix}. \tag{95}$$

(F2) *if* $n_1^* \geq n_{01}$ *and* $n_2^* < n_{02}$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1(n_1^*, T_{1,n_1^*}) \\ \prod_1(n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_2(n_{02}, T_{2,n_{02}}) \\ \prod_3(1, T_{3,1}) \end{Bmatrix}. \tag{96}$$

(F3) *if* $n_1^* < n_{01}$ *and* $n_{02} \leq n_2^* < n_{01} - 1$, *then*

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1(n_{01}, T_{1,n_{01}}) \\ \prod_2(n_2^*, T_{2,n_2^*}) \\ \prod_2(n_2^* + 1, T_{2,n_2^*+1}) \\ \prod_3(1, T_{3,1}) \end{Bmatrix}. \tag{97}$$

(F4) *if* $n_1^* \geq n_{01}$ *and* $n_{02} \leq n_2^* < n_{01} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_2 (n_2^*, T_{2,n_2^*}) \\ \prod_2 (n_2^* + 1, T_{2,n_2^*+1}) \\ \prod_3 (1, T_{3,1}) \end{array} \right\}. \tag{98}$$

(F5) *if* $n_1^* < n_{01}$ *and* $n_2^* \geq n_{01} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (n_{01} - 1, T_{2,n_{01}-1}) \\ \prod_3 (1, T_{3,1}) \end{array} \right\}. \tag{99}$$

(F6) *if* $n_1^* \geq n_{01}$ *and* $n_2^* \geq n_{01} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_2 (n_{01} - 1, T_{2,n_{01}-1}) \\ \prod_3 (1, T_{3,1}) \end{array} \right\}. \tag{100}$$

(S-2): $n_{01} = n_{02}$.

(A) *If* $P_1 \leq 0$ *and* $P_3 \leq 0$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_3 (1, T_{1,3}) \end{array} \right\}. \tag{101}$$

(B) *Suppose that* $P_1 > 0$ *and* $P_3 > 0$. *Hence,*

(B1) *if* $n_1^* < n_{01}$ *and* $n_3^* < n_{01} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_3 (n_3^*, T_{3,n_3^*}) \\ \prod_3 (n_3^* + 1, T_{3,n_3^*+1}) \end{array} \right\}. \tag{102}$$

(B2) *if* $n_1^* \geq n_{01}$ *and* $n_3^* < n_{01} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_3 (n_3^*, T_{3,n_3^*}) \\ \prod_3 (n_3^* + 1, T_{3,n_3^*+1}) \end{array} \right\}. \tag{103}$$

(B3) *if* $n_1^* < n_{01}$ *and* $n_3^* \geq n_{01} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_3 (n_{01} - 1, T_{3,n_{01}-1}) \end{array} \right\}. \tag{104}$$

(B4) *if* $n_1^* \geq n_{01}$ *and* $n_3^* \geq n_{01} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_3 (n_{01} - 1, T_{3,n_{01}-1}) \end{array} \right\}. \tag{105}$$

(C) *Suppose that* $P_1 \leq 0$ *and* $P_3 > 0$. *Hence,*
(C1) *if* $n_3^* < n_{01} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_3 (n_3^*, T_{3,n_3^*}) \\ \prod_3 (n_3^* + 1, T_{3,n_3^*+1}) \end{array} \right\}. \tag{106}$$

(C2) *if* $n_3^* \geq n_{01} - 1$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_3 (n_{01} - 1, T_{3,n_{01}-1}) \end{array} \right\}. \tag{107}$$

(D) *Suppose that* $P_1 > 0$ *and* $P_3 \leq 0$.
(D1) *if* $n_1^* < n_{01}$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_3 (1, T_{3,1}) \end{array} \right\}. \tag{108}$$

(D2) *if* $n_1^* \geq n_{01}$, *then*

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \\ \prod_3 (1, T_{3,1}) \end{array} \right\}. \tag{109}$$

Proof. *Case* (I): suppose that $G_1(1) \geq 0$. Since $G_1(n)$ and $G_2(n)$ are increasing on $n \geq 1$, we have $G_2(n) \geq G_1(n) \geq G_1(1)$. By Theorem 1, we get

$$T^{(n)} = T_{1,n} \quad \text{for all } n \geq 1, \tag{110}$$

and

$$\prod(n, T^{(n)}) = \prod_1 (n, T_{1,n}) \quad \text{for all } n \geq 1. \tag{111}$$

(A) If $P_1 \leq 0$, Lemma 3(IA) implies that $TC_1(n) = \prod_1(n, T_{1,n})$ is decreasing on $n \geq 1$. Eqs. (26), (36), (37) and (111) reveal that

$$(n^*, T^*) = (1, T_{1,1}) = \left(1, \sqrt{\frac{\frac{2R}{D}(S_1 + S_2 + F)}{R(vr_2 + pI_{2e}) + AD}}\right),$$

and

$$\prod(n^*, T^*) = \prod(1, T_{1,1}) = \max_{n \geq 1} \left\{ \prod(n, T^{(n)}) \right\}.$$

(B) If $P_1 > 0$, Lemma 3(IB) implies that $TC_1(n) = \prod(n, T_{1,n})$ is increasing on $(0, E_1]$ and decreasing on $[E_1, \infty)$. Then, $TC_1(n) = \prod_1(n, T_{1,n})$ has the maximum value at n_1^* or $n_1^* + 1$ where $n_1^* = \lfloor E_1 \rfloor$. According to Eqs. (36), (37) and (102), we have

$$\prod(n^*, T^*) = \max \left\{ \begin{array}{l} \prod_1 (n_1^*, T_{1,n_1^*}) \\ \prod_1 (n_1^* + 1, T_{1,n_1^*+1}) \end{array} \right\}.$$

Case (II): suppose that $G_1(1) < 0$ and $G_2(1) \geq 0$. Since $\lim_{n\to\infty} G_1(n) = \infty$, there exists the smallest positive integer n_{01} such that $G_1(n_{01}) \geq 0$ and $n_{01} > 1$. So, we have

$$\text{(a) } G_2(n) \geq G_1(n) \geq G_1(n_{01}) \geq 0 \quad \text{if } n \geq n_{01}, \tag{112}$$

and

$$\text{(b) } G_1(n) < 0 \quad \text{if } 1 \leq n \leq n_{01} - 1. \tag{113}$$

Furthermore, if $G_2(1) \geq 0$, then

$$G_2(n) \geq G_2(1) \geq 0 > G_1(n) \quad \text{if } 1 \leq n \leq n_{01} - 1. \tag{114}$$

According to Eqs. (112)–(114), Theorem 1(A) and (B) imply

$$\text{(c) } T^{(n)} = T_{1,n} \quad \text{if } n \geq n_{01}, \tag{115}$$

and

$$\text{(d) } T^{(n)} = T_{2,n} \quad \text{if } 1 \leq n \leq n_{01} - 1. \tag{116}$$

Eqs. (115) and (116) reveal

$$\prod(n, T^{(n)}) = \begin{cases} \prod_1 (n, T_{1,n}) & \text{if } n \geq n_{01}, & \text{(a)} \\ \prod_2 (n, T_{2,n}) & \text{if } 1 \leq n \leq n_{01} - 1. & \text{(b)} \end{cases} \tag{117}$$

(A) If $P_1 \leq 0$ and $P_2 \leq 0$, then Lemma 3(IA) and (IIA) imply that
  (i) $TC_1(n) = \prod_1(n, T_{1,n})$ is decreasing on $n \geq n_{01}$,
  (ii) $TC_2(n) = \prod_2(n, T_{2,n})$ is decreasing on $1 \leq n \leq n_{01} - 1$.
  Eqs. (36), (37), (117)(a) and (b) and (i)–(ii) show that

$$\prod(n^*, T^*) = \max \begin{Bmatrix} \prod_1 (n_{01}, T_{1,n_{01}}) \\ \prod_2 (1, T_{2,1}) \end{Bmatrix}.$$

(B) Suppose that $P_1 > 0$ and $P_2 > 0$. Hence,
  (B1) if $n_1^* < n_{01}$ and $n_2^* < n_{01} - 1$, then Lemma 3(IA) and (IIB) imply that
    (b11) $TC_1(n)$ is decreasing on $n \geq n_{01}$,
    (b12) $TC_2(n)$ is increasing on $(0, E_2]$ and decreasing on $[E_2, n_{01} - 1]$.
    Eqs. (36), (37), (117)(a) and (b) and (b11)–(b12) show that Eq. (63) holds.
  (B2) if $n_1^* \geq n_{01}$ and $n_2^* < n_{01} - 1$, then Lemma 3(IB) and (IIB) imply that
    (b21) $TC_1(n)$ is increasing on $[n_{01}, E_1]$ and decreasing on $[E_1, \infty)$,
    (b22) $TC_2(n)$ is increasing on $(0, E_2]$ and decreasing on $[E_2, n_{01} - 1]$.
    Eqs. (36), (37), (117)(a) and (b) and (b21)–(b22) show that Eq. (64) holds.
  (B3) if $n_1^* < n_{01}$ and $n_2^* \geq n_{01} - 1$, then Lemma 3(IA) and (IIB) imply that
    (b31) $TC_1(n)$ is decreasing on $n \geq n_{01}$,
    (b32) $TC_2(n)$ is increasing on $1 \leq n \leq n_{01} - 1$.
    Eqs. (36), (37), (117)(a) and (b) and (b31)–(b32) show that Eq. (65) holds.
  (B4) if $n_1^* \geq n_{01}$ and $n_2^* \geq n_{01} - 1$, then Lemma 3(IB) and (IIB) imply that
    (b41) $TC_1(n)$ is increasing on $[n_{01}, E_1]$ and decreasing on $[E_1, \infty)$.
    (b42) $TC_2(n)$ is increasing on $1 \leq n \leq n_{01} - 1$.
    Eqs. (36), (37), (117)(a) and (b) and (b41)–(b42) show that Eq. (66) holds.
(C) Suppose that $P_1 \leq 0$ and $P_2 > 0$. Incorporating the above ways of arguments of (A) and [B:B1, B2, B3, B4], with Lemma 3, we can demonstrate that:
  (C1) if $n_2^* < n_{01} - 1$, then Eq. (67) holds.
  (C2) if $n_2^* \geq n_{01} - 1$, then Eq. (68) holds.
(D) Suppose that $P_1 > 0$ and $P_2 \leq 0$. Incorporating the above ways of arguments of (A) and [B:B1, B2, B3, B4], with Lemma 3, we can demonstrate that:
  (D1) if $n_1^* < n_{01}$, then Eq. (69) holds.
  (D2) if $n_1^* \geq n_{01}$, then Eq. (70) holds.

Case (III): suppose that $G_1(1) < 0$ and $G_2(1) < 0$. Since $\lim_{n\to\infty} G_2(n) = \infty$, there exists the smallest positive integer n_{02} such that $G_2(n_{02}) \geq 0$ and $n_{01} \geq n_{02} > 1$. So, $G_2(n) < 0$ if $1 \leq n \leq n_{02} - 1$. We also have

$$0 > G_2(n) \geq G_1(n) \quad \text{if } 1 \leq n \leq n_{02} - 1, \tag{118}$$

Then, there are two situations (S-1): $n_{01} > n_{02}$ and $(\text{S - 2}) : n_{01} = n_{02}$.
(S-1): $n_{01} > n_{02}$.
Under this situation,

$$G_2(n) \geq 0 > G_1(n) \quad \text{if } n_{02} \leq n \leq n_{01} - 1. \tag{119}$$

With Eqs. (109) and (110), Theorem 1(B) and (C) imply

$$T^{(n)} = T_{2,n} \quad \text{if } n_{02} \leq n \leq n_{01} - 1, \tag{120}$$

$$T^{(n)} = T_{3,n} \quad \text{if } 1 \leq n \leq n_{02} - 1. \tag{121}$$

Combining Eqs. (115), (120) and (121), we get

$$\prod(n, T^{(n)}) = \begin{cases} \prod_3(n, T_{3,n}) & \text{if } 1 \leq n \leq n_{02} - 1, & \text{(a)} \\ \prod_2(n, T_{2,n}) & \text{if } n_{02} \leq n \leq n_{01} - 1, & \text{(b)} \\ \prod_1(n, T_{1,n}) & \text{if } n \geq n_{01}. & \text{(c)} \end{cases} \tag{122}$$

(S-2): $n_{01} = n_{02}$.
Under this situation,

$$G_2(n) > G_1(n) \geq 0 \quad \text{if } n \geq n_{01} = n_{02}. \tag{123}$$

With Eq. (123), Theorem 1(A) implies

$$T^{(n)} = T_{1,n} \quad \text{if } n \geq n_{01} = n_{02}. \tag{124}$$

Combining (118) and (124), we get

$$\prod(n, T^{(n)}) = \begin{cases} \prod_3(n, T_{3,n}) & \text{if } 1 \leq n \leq n_{01}, & \text{(a)} \\ \prod_1(n, T_{1,n}) & \text{if } n \geq n_{01}. & \text{(b)} \end{cases} \tag{125}$$

About (S-1): $n_{01} > n_{02}$.

(A) If $P_1 \leq 0$ and $P_3 < P_2 \leq 0$, then Lemma 3(IA), (IIA) and (IIIA) imply that
  (a1) $TC_1(n)$ is decreasing on $n \geq n_{01}$,
  (a2) $TC_2(n)$ is decreasing on $n_{02} \leq n \leq n_{01} - 1$,
  (a3) $TC_3(n)$ is decreasing on $1 \leq n \leq n_{02} - 1$.
  Eqs. (36), (37), (122)(a)–(c) and (a1)–(a3) show that Eq. (71) holds.
(B) Suppose that $P_1 > 0$ and $P_2 > P_3 > 0$. Hence,
  (B1) if $n_1^* < n_{01}$, $n_2^* < n_{02}$ and $n_3^* < n_{02} - 1$, then Lemma 3(IA), (IIA) and (IIIB) imply that
    (b1a) $TC_1(n)$ is decreasing on $[n_{01}, \infty)$,
    (b1b) $TC_2(n)$ is decreasing on $[n_{02}, n_{01} - 1]$,
    (b1c) $TC_3(n)$ is increasing on $(0, E_3]$ and decreasing on $[E_3, n_{02} - 1]$.
    Eqs. (36), (37), (122)(a)–(c) and (b1a)–(b1c) show that Eq. (72) holds.
  (B2) if $n_1^* \geq n_{01}$, $n_2^* < n_{02}$ and $n_3^* < n_{02} - 1$, then Lemma 3(IB), (IIA) and (IIIB) imply that
    (b2a) $TC_1(n)$ is increasing on $[n_{01}, E_1]$ and decreasing on $[E_1, \infty)$.
    (b2b) $TC_2(n)$ is decreasing on $[n_{02}, n_{01} - 1]$,
    (b2c) $TC_3(n)$ is increasing on $(0, E_3]$ and decreasing on $[E_3, n_{02} - 1]$.
    Eqs. (36), (37), (122)(a)–(c) and (b2a)–(b2c) show that Eq. (73) holds.
  (B3) if $n_1^* < n_{01}$, $n_{02} \leq n_2^* < n_{01} - 1$ and $n_3^* < n_{02} - 1$, then Lemma 3(IA), (IIB) and (IIIB) imply that
    (b3a) $TC_1(n)$ is decreasing on $[n_{01}, \infty)$,
    (b3b) $TC_2(n)$ is increasing on $[n_{02}, E_2]$ and decreasing on $[E_2, n_{01} - 1]$,
    (b3c) $TC_3(n)$ is increasing on $(0, E_3]$ and decreasing on $[E_3, n_{02} - 1]$.
    Eqs. (36), (37), (122)(a)–(c) and (b3a)–(b3c) show that Eq. (74) holds.

(B4) if $n_1^* \geq n_{01}$, $n_{02} \leq n_2^* < n_{01} - 1$ and $n_3^* < n_{02} - 1$, then Lemma 3(IB), (IIB) and (IIIB) imply that
(b4a) $TC_1(n)$ is increasing on $[n_{01}, E_1]$ and decreasing on $[E_1, \infty)$,
(b4b) $TC_2(n)$ is increasing on $[n_{02}, E_2]$ and decreasing $[E_2, n_{01} - 1]$,
(b4c) $TC_3(n)$ is increasing on $(0, E_3]$ and decreasing on $[E_3, n_{02} - 1]$.
Eqs. (36), (37), (122)(a)–(c) and (b4a)–(b4c) show that Eq. (75) holds.

(B5) if $n_1^* < n_{01}$, $n_2^* \geq n_{01} - 1$ and $n_3^* < n_{02} - 1$, then Lemma 3(IA), (IIB) and (IIIB) imply that
(b5a) $TC_1(n)$ is decreasing on $[n_{01}, \infty)$,
(b5b) $TC_2(n)$ is increasing on $[n_{02}, n_{01} - 1]$,
(b5c) $TC_3(n)$ is increasing on $(0, E_3]$ and decreasing on $[E_3, n_{02} - 1]$.
Eqs. (36), (37), (122)(a)–(c) and (b5a)–(b5c) show that Eq. (76) holds.

(B6) if $n_1^* \geq n_{01}$, $n_2^* \geq n_{01} - 1$ and $n_3^* < n_{02} - 1$, then Lemma 3(IB), (IIB) and (IIIB) imply that
(b6a) $TC_1(n)$ is increasing on $[n_{01}, E_1]$ and decreasing on $[E_1, \infty)$,
(b6b) $TC_2(n)$ is increasing on $[n_{02}, n_{01} - 1]$,
(b6c) $TC_3(n)$ is increasing on $(0, E_3]$ and decreasing on $[E_3, n_{02} - 1]$.
Eqs. (36), (37), (122)(a)–(c) and (b6a)–(b6c) show that Eq. (77) holds.

(B7) if $n_1^* < n_{01}$, $n_2^* < n_{02}$ and $n_3^* \geq n_{02} - 1$, then Lemma 3(IA), (IIA) and (IIIB) imply that
(b7a) $TC_1(n)$ is decreasing on $[n_{01}, \infty)$,
(b7b) $TC_2(n)$ is decreasing on $[n_{02}, n_{01} - 1]$,
(b7c) $TC_3(n)$ is increasing on $(0, n_{02} - 1]$.
Eqs. (36), (37), (122)(a)–(c) and (b7a)–(b7c) show that Eq. (78) holds.

(B8) if $n_1^* \geq n_{01}$, $n_2^* < n_{02}$ and $n_3^* \geq n_{02} - 1$, then Lemma 3(IB), (IIA) and (IIIB) imply that
(b8a) $TC_1(n)$ is increasing on $[n_{01}, E_1]$ and decreasing on $[E_1, \infty)$,
(b8b) $TC_2(n)$ is decreasing on $[n_{02}, n_{01} - 1]$,
(b8c) $TC_3(n)$ is increasing on $(0, n_{02} - 1]$.
Eqs. (36), (37), (122)(a)–(c) and (b8a)–(b8c) show that Eq. (79) holds.

(B9) if $n_1^* < n_{01}$, $n_{02} \leq n_2^* < n_{01} - 1$ and $n_3^* \geq n_{02} - 1$, then Lemma 3(IA), (IIB) and (IIIB) imply that
(b9a) $TC_1(n)$ is decreasing on $[n_{01}, \infty)$,
(b9b) $TC_2(n)$ is increasing on $[n_{02}, E_2]$ and decreasing on $[E_2, n_{01} - 1]$,
(b9c) $TC_3(n)$ is increasing on $(0, n_{02} - 1]$.
Eqs. (36), (37), (122)(a)–(c) and (b9a)–(b9c) show that Eq. (80) holds.

(B10) if $n_1^* \geq n_{01}$, $n_{02} \leq n_2^* < n_{01} - 1$ and $n_3^* \geq n_{02} - 1$, then Lemma 3(IB), (IIB) and (IIIB) imply that
(b10a) $TC_1(n)$ is increasing on $[n_{01}, E_1]$ and decreasing on $[E_1, \infty)$,
(b10b) $TC_2(n)$ is increasing on $[n_{02}, E_2]$ and decreasing on $[E_2, n_{01} - 1]$,
(b10c) $TC_3(n)$ is increasing on $(0, n_{02} - 1]$.
Eqs. (36), (37), (122)(a)–(c) and (b10a)–(b10c) show that Eq. (81) holds.

(B11) if $n_1^* < n_{01}$, $n_2^* \geq n_{01} - 1$ and $n_3^* \geq n_{02} - 1$, the Lemma 3(IA), (IIB) and (IIIB) imply that
(b11a) $TC_1(n)$ is decreasing on $[n_{01}, \infty)$,
(b11b) $TC_2(n)$ is increasing on $[n_{02}, n_{01} - 1]$,
(b11c) $TC_3(n)$ is increasing on $(0, n_{02} - 1]$.
Eqs. (36), (37), (122)(a)–(c) and (b11a)–(b11c) show that Eq. (82) holds.

(B12) if $n_1^* \geq n_{01}$, $n_2^* \geq n_{01} - 1$ and $n_3^* \geq n_{02} - 1$, then Lemma 3(IB), (IIB) and (IIIB) imply that
(b12a) $TC_1(n)$ is increasing on $[n_{01}, E_1]$ and decreasing on $[E_1, \infty)$,
(b12b) $TC_2(n)$ is increasing on $[n_{02}, n_{01} - 1]$,
(b12c) $TC_3(n)$ is increasing on $(0, n_{02} - 1]$.
Eqs. (36), (37), (122)(a)–(c) and (b12a)–(b12c) show that Eq. (83) holds.

(C) Suppose that $P_1 \leq 0$, $P_2 > 0$ and $P_3 \leq 0$. Incorporating the above ways of arguments of (A) and [B:B1-B12], with Lemma 3, we can demonstrate that
(C1) if $n_2^* < n_{02}$, then Eq. (84) holds.
(C2) if $n_{02} \leq n_2^* < n_{01} - 1$, then Eq. (85) holds
(C3) if $n_2^* \geq n_{01} - 1$, then Eq. (86) holds.

(D) Suppose that $P_1 \leq 0$ and $P_2 > P_3 > 0$. Incorporating the above ways of arguments of (A) and [B:B1-B12], with Lemma 3, we can demonstrate that
(D1) if $n_2^* < n_{02}$ and $n_3^* < n_{02} - 1$, then Eq. (87) holds.
(D2) if $n_{02} \leq n_2^* < n_{01} - 1$ and $n_3^* < n_{02} - 1$, then Eq. (88) holds.
(D3) if $n_2^* \geq n_{01} - 1$ and $n_3^* < n_{02} - 1$, then Eq. (89) holds.
(D4) if $n_2^* < n_{02}$ and $n_3^* \geq n_{02} - 1$, then Eq. (90) holds.
(D5) if $n_{02} \leq n_2^* < n_{01} - 1$ and $n_3^* \geq n_{02} - 1$, then Eq. (91) holds.
(D6) if $n_2^* \geq n_{01} - 1$ and $n_3^* \geq n_{02} - 1$, then Eq. (92) holds.

(E) Suppose that $P_1 > 0$ and $P_3 < P_2 \leq 0$. Incorporating the above ways of arguments of (A) and [B:B1-B12], with Lemma 3, we can demonstrate that
(E1) if $n_1^* < n_{01}$, then Eq. (93) holds.

Table 1
Computation results for values when $F = 0$.

t_1	t_2	δ	$G_1(1)$	$G_2(1)$	n_{01}	n_{02}	P_1	P_2	P_3	n^*	T^*	$\prod_{(n^*,T^*)}$ (*Profit*)	Equation[a]
30	0	0	<0	<0	5	5	>0	>0	>0	$n_1^* = 5$	$T_{1,n^*} = 28.24$	$\prod_1(n^*, T^*) = 11080$	105
30	10	0	<0	<0	8	5	>0	>0	>0	$n_2^*+1 = 6$	$T_{2,n^*} = 24.13$	$\prod_2(n^*, T^*) = 10863$	80
30	10	1	<0	<0	8	5	>0	>0	>0	$n_2^*+1 = 6$	$T_{2,n^*} = 23.90$	$\prod_2(n^*, T^*) = 11212$	80
30	10	1.5	<0	<0	8	5	>0	>0	>0	$n_2^*+1 = 6$	$T_{2,n^*} = 23.78$	$\prod_2(n^*, T^*) = 11392$	80
30	10	2	<0	<0	8	5	>0	>0	>0	$n_2^*+1 = 6$	$T_{2,n^*} = 23.67$	$\prod_2(n^*, T^*) = 11572$	80
30	0	1.5	<0	<0	5	5	>0	>0	>0	$n_1^* = 5$	$T_{1,n^*} = 28.24$	$\prod_1(n^*, T^*) = 11080$	105
30	20	1.5	<0	<0	20	5	>0	>0	>0	$n_2^* = 6$	$T_{2,n^*} = 22.72$	$\prod_2(n^*, T^*) = 11668$	80
60	0	1.5	<0	<0	2	2	>0	>0	>0	$n_1^* = 5$	$T_{1,n^*} = 28.24$	$\prod_1(n^*, T^*) = 10875$	105
60	10	1.5	<0	<0	3	2	>0	>0	>0	$n_1^* = 5$	$T_{1,n^*} = 27.79$	$\prod_1(n^*, T^*) = 11187$	83
60	20	1.5	<0	<0	3	2	>0	>0	>0	$n_1^* = 5$	$T_{1,n^*} = 27.35$	$\prod_1(n^*, T^*) = 11504$	83
60	30	1.5	<0	<0	5	2	>0	>0	>0	$n_1^* = 5$	$T_{1,n^*} = 26.92$	$\prod_1(n^*, T^*) = 11827$	83
60	40	1.5	<0	<0	8	2	>0	>0	>0	$n_2^*+1 = 6$	$T_{2,n^*} = 22.81$	$\prod_2(n^*, T^*) = 12155$	80
60	50	1.5	<0	<0	20	2	>0	>0	>0	$n_2^*+1 = 7$	$T_{2,n^*} = 19.55$	$\prod_2(n^*, T^*) = 12443$	80
90	0	1.5	<0	<0	2	2	>0	>0	>0	$n_1^* = 5$	$T_{1,n^*} = 28.24$	$\prod_1(n^*, T^*) = 10669$	105
90	10	1.5	<0	>0	2	1	>0	>0	>0	$n_1^* = 5$	$T_{1,n^*} = 27.79$	$\prod_1(n^*, T^*) = 10972$	66
90	20	1.5	<0	>0	2	1	>0	>0	>0	$n_1^* = 5$	$T_{1,n^*} = 27.35$	$\prod_1(n^*, T^*) = 11281$	66
90	30	1.5	<0	>0	2	1	>0	>0	>0	$n_1^* = 5$	$T_{1,n^*} = 26.92$	$\prod_1(n^*, T^*) = 11595$	66
90	40	1.5	<0	>0	3	1	>0	>0	>0	$n_1^*+1 = 6$	$T_{1,n^*} = 23.15$	$\prod_1(n^*, T^*) = 11916$	66
90	50	1.5	<0	>0	3	1	>0	>0	>0	$n_1^*+1 = 6$	$T_{1,n^*} = 22.82$	$\prod_1(n^*, T^*) = 12243$	66
90	60	1.5	<0	>0	5	1	>0	>0	>0	$n_1^*+1 = 6$	$T_{1,n^*} = 22.51$	$\prod_1(n^*, T^*) = 12574$	66
90	70	1.5	<0	>0	7	1	>0	>0	>0	$n_{01}^{-}1 = 6$	$T_{2,n^*} = 21.95$	$\prod_2(n^*, T^*) = 12911$	65
90	80	1.5	<0	>0	19	1	>0	>0	>0	$n_2^* = 7$	$T_{2,n^*} = 18.80$	$\prod_2(n^*, T^*) = 13209$	63

[a] Equation: which equation is used to determine the optimal solution?

(E2) if $n_1^* \geq n_{01}$, then Eq. (94) holds.

(F) Suppose that $P_1 > 0, P_2 > 0$ and $P_3 \leq 0$. Incorporating the above ways of arguments of (A) and [B:B1-B12], with Lemma 3, we can demonstrate that
(F1) if $n_1^* < n_{01}$ and $n_2^* < n_{02}$, then Eq. (95) holds.
(F2) if $n_1^* \geq n_{01}$ and $n_2^* < n_{02}$, then Eq. (96) holds.
(F3) if $n_1^* < n_{01}$ and $n_{02} \leq n_2^* < n_{01} - 1$, then Eq. (97) holds.
(F4) if $n_1^* \geq n_{01}$ and $n_{02} \leq n_2^* < n_{01} - 1$, then Eq. (98) holds.
(F5) if $n_1^* < n_{01}$ and $n_2^* \geq n_{01} - 1$, then Eq. (99) holds.
(F6) if $n_1^* \geq n_{01}$ and $n_2^* \geq n_{01} - 1$, then Eq. (100) holds.

About (S-2): $n_{01} = n_{02}$. Incorporating the above ways of arguments of (A), [B:B1-B12], [C:C1-C3], [D:D1-D6], [E:E1-E2] and [F:F1-F6], with Eqs. (125)(a) and (b) and Lemma 3, we can demonstrate that Eqs. (101)–(109) hold.

Combining all arguments about Cases (I)–(III), we have completed the proof of Theorem 2. □

7. Comparisons with results of [36]

The following examples in Table 1 are used to compare the solution procedure in this paper with that of [36]. We assume that $R = 3200$ units/year, $D_0 = 1000$ units/year, $v = \$20$/unit, $p = \$25$/unit, $S_1 = \$400$/setup, $S_2 = \$25$/order, $r_1 = 0.2$, $r_2 = 0.2$, $c = \$11$/unit, $I_{1p} = 0.5$, $I_{2p} = 0.5$, $I_{2e} = 0.3$ and $F = 0$. Following Theorem 2 in this paper, we obtain Table 1. All optimal solutions of examples in Table 1 are consistent with the corresponding those of [36]. They reveal that Theorem 2 is rather accurate.

8. Conclusions

To make more real situations, this paper takes the transportation cost into account to develop the new integrated supplier–retailer inventory model under the condition that both supplier and retailer have adopted the two-level trade credit policy for generalizing [36]. This paper uses the calculus approach not only to derive the closed-form formulations for the optimal solution but also to simplify the algorithm to locate the optimal solution described in [36]. Based on the above arguments, we have the following observations:

(C1) When $F = 0$, if $H_1(n) \le 0$ and $H_2(n) \le 0$, then the inside numbers in both radicals of $T_{2,n}$ and $T_{3,n}$ are negative or 0. So, both $T_{2,n}$ and $T_{3,n}$ do not exist if $H_1(n) \le 0$ and $H_2(n) \le 0$. Therefore, validities of Theorem 4.1 and the algorithm to locate the optimal solution in [36] are based on whether both $H_1(n)$ and $H_2(n)$ are more than 0. However, Lemma 2(A) and (B) in this paper complement those shortcomings of non-existences of both $T_{2,n}$ and $T_{3,n}$ in Theorem 4.1 and the algorithm to locate the optimal solution in [36] from the viewpoint of logic. Furthermore, Theorem 1 in this paper is valid without any restrictions on both $H_1(n)$ and $H_2(n)$ to improve Theorem 4.1 in [36].

(C2) When $F = 0$, this paper makes numerical comparisons with [36]. Table 1 illustrates that the closed-form formulations for the optimal solution are rather accurate.

Incorporating (C1) and (C2), we conclude that this paper improves [36].

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